ON THE CONVERGENCE OF SERIES DETERMINING THE BOUNDARIES OF THE REGIONS OF INSTABILITY OF SOLUTIONS OF A SECOND ORDER LINEAR DIFFERENTIAL EQUATION WITH PERIODIC COEFFICIENTS

(O SKHODIMOSTI RIADOV, OPREDELIAIUSHCHIKH GRANITSY OBLASTEI NEUSTOICHIVOSTI RESHENII LINEINOGO DIFFERENTSIAL'NOGO URAVNENIIA VTOROGO PORIADKA S PERIODICHESKIMI KOEFFITSIENTAMI)

PMM Vol.27, No.3, 1963, pp.565-572

K. G. VALEEV
(Leningrad)

(Received October 15, 1962)

1. The following differential equation is considered

$$\frac{d^2y}{dt^2} + (\lambda - \mu a(t))\, y = 0 \tag{1.1}$$

Here $a(t)$ is a real periodic function of t, with period π determined by the expansion

$$a(t) = \sum_{s=-\infty}^{\infty} a_s e^{2ist}, \qquad a_{-s} = \bar{a}_s, \qquad \operatorname{Im} a_0 = 0 \tag{1.2}$$

Let us assume that some of the first few terms of the expansions determining the boundary of the region of instability on the plane of the parameters μ, λ have been found by the method of a small parameter ([1], p.321). Here $\lambda_{n1}(\mu) \geqslant \lambda_{n2}(\mu)$

$$\begin{aligned} \lambda_{n1}(\mu) &= n^2 + \mu b_1 + \mu^2 b_2 + \ldots + \mu^r b_r + \varepsilon_1(\mu) \\ \lambda_{n2}(\mu) &= n^2 + \mu c_1 + \mu^2 c_2 + \ldots + \mu^r c_r + \varepsilon_2(\mu) \end{aligned} \qquad (n = 0, 1, 2, \ldots) \tag{1.3}$$

We shall attempt to obtain estimates of the functions $\varepsilon_1(\mu)$ and $\varepsilon_2(\mu)$ in terms of the coefficients a_s of the Fourier series (1.2) and of some already known coefficients b_s and c_s of the expansion (1.3).

2. On the boundary of the region of instability of the equation (1.1) there exists a periodic solution of period 2π and of the form [1]

$$y = e^{nit} \sum_{k=-\infty}^{\infty} e^{2ikt} y_k \tag{2.1}$$

Substituting (2.1) into the equation (1.1), we find an infinite system of linear algebraic equations for the determination of the quantities y_k

$$[\lambda - (n + 2k)^2] y_k - \mu \sum_{s=-\infty}^{\infty} a_{k-s} y_s = 0 \qquad (k = 0, \pm 1, \pm 2, \ldots) \tag{2.2}$$

The condition for the existence of a nontrivial periodic solution of the equation (1.1) of period 2π yields an equation that determines the boundary of the region of instability. One can obtain this condition in explicit form if one equates to zero the infinite determinant of Hill. The latter can be transformed into a finite order determinant [2]. A more natural procedure is the use of a method ([3], pp. 164 to 168) used already earlier for the equation (1.1) in the work [4]. This method is employed below, and it leads to an equation that coincides with an equation of the work [2].

For the time being, let us exclude from our consideration the two equations of (2.2) for which $k = 0$, and $-n$ (when $n = 0$, we exclude only one equation with $k = 0$). The quantities $y_k (k \neq 0, -n)$ shall be expressed by means of complex-conjugate quantities y_0, y_{-n}. We substitute the obtained expressions for $y_k (k \neq 0, -n)$ into the remaining two equations (when $n = 0$ only one equation) in order to find y_0 and y_{-n}. The condition for the existence of a non-zero solution y_0 determines the boundary of the region of instability.

3. Let us consider the zero region of instability $n = 0$. We introduce the notation

$$d_n (k) = [\lambda + (n + 2k)^2]^{-1} \tag{3.1}$$

When $k \neq 0$, the equations (2.2) can be written in the form

$$y_k = \mu \sum_{s=-\infty}^{\infty}{}' d_0 (k) a_{k-s} y_s + \mu d_0 (k) a_k y_0 \qquad (k = \pm 1, \pm 2, \ldots) \tag{3.2}$$

The prime ($'$) on the sum indicates here and in what follows that the terms with the indices 0, $-n$ (here $-n = 0$) are omitted from the sum.

By the method of successive approximations ([3], p. 160) we obtain

$$y_k = \left[\mu d_0(k)\, a_k + \mu^2 \sum_{\alpha=-\infty}^{\infty}{}' d_0(k)\, a_{k-\alpha}\, d_0(\alpha)\, a_k + \ldots\right] y_0 \tag{3.3}$$

Substituting (3.3) into (2.2) with $k = 0$ and dividing by $y_0 \neq 0$, we obtain

$$\lambda = \mu a_0 + \mu^2 \sum_{\alpha=-\infty}^{\infty}{}' a_{-\alpha}\, d_0(\alpha)\, a_\alpha + \mu^3 \sum_{\alpha,\,\beta=-\infty}^{\infty}{}' a_{-\alpha}\, d_0(\alpha)\, a_{\alpha-\beta}\, d_0(\beta)\, a_\beta +$$
$$+ \mu^4 \sum_{\alpha,\,\beta,\,\gamma=-\infty}^{\infty}{}' a_{-\alpha}\, d_0(\alpha)\, a_{\alpha-\beta}\, d_0(\beta)\, a_{\beta-\gamma}\, d_0(\gamma)\, a_\gamma + \ldots \equiv \Psi(\mu, \lambda) \tag{3.4}$$

4. From the equation (3.4) one can obtain $\lambda_0(\mu)$ in the form of a power series in μ. Let us construct a dominating series (majorant) for the right part $\Psi(\mu, \lambda)$ of the equation (3.4) of the work [5].

Suppose that the power series in μ of $f_1(\mu)$ (or the series for $\varphi_1(\mu, \lambda)$ in powers of μ and λ) is dominated by the series for $f_2(\mu)$ (or $\varphi_2(\mu, \lambda)$). We will denote this by writing

$$f_1(\mu) \ll f_2(\mu), \qquad \varphi_1(\mu, \lambda) \ll \varphi_2(\mu, \lambda) \tag{4.1}$$

Let us introduce the notation

$$\gamma_1 = \sum_{\substack{s=-\infty \\ s\neq 0}}^{\infty} |a_s|, \qquad \gamma_2 = \max_s |a_s|, \qquad q(\lambda) = (4 - \lambda)^{-1} \tag{4.2}$$

if $|\lambda| < 4$ we obtain for $d_0(k)$ in (3.1) the relation

$$d_0(k) \equiv (\lambda - 4k^2)^{-1} \ll (4 - \lambda)^{-1} = q(\lambda) \qquad (k = \pm 1, \pm 2, \ldots) \tag{4.3}$$

If one considers the system of equations (3.2) in the linear normed space m, then the system of equations (3.2) will be entirely regular ([3], pp. 67,167) when $\mu\gamma_1 q(\lambda) < 1$. Hereby the right-hand side of (3.4) $\Psi(\mu, \lambda)$ is dominated by the series

$$\Psi(\mu, \lambda) \ll \mu\gamma_2 + \mu\gamma_2(\mu\gamma_1 q) + \mu\gamma_2(\mu\gamma_1 q)^2 + \ldots = \mu\gamma_2(1 - \mu\gamma_1 q)^{-1} \tag{4.4}$$

Solving the equation

$$\lambda = \mu\gamma_2\,[1 - \mu\gamma_1(4 - \lambda)^{-1}]^{-1} \tag{4.5}$$

for λ, we obtain the function $\lambda(\mu)$ which dominates [5] the solution

$\lambda_0(\mu)$ of the equation (3.4)

$$\lambda(\mu) = 2 + 0.5\mu(\gamma_2 - \gamma_1) - \sqrt{4 - 2\mu(\gamma_1 + \gamma_2) + 0.25\mu^2(\gamma_1 - \gamma_2)^2} =$$
$$= 2 + 0.5\mu(\gamma_2 - \gamma_1) - 2\sqrt{[1 - 0.25\mu(\sqrt{\gamma_1} + \sqrt{\gamma_2})^2]\,[1 - 0.25\mu(\sqrt{\gamma_1} - \sqrt{\gamma_2})^2]} \quad (4.6)$$

Let us introduce the notation, which makes use of the quantities γ_1, γ_2 determined by (4.2)

$$h_1 = 0.25(\sqrt{\gamma_1} + \sqrt{\gamma_2})^2, \qquad h_2 = 0.25(\sqrt{\gamma_1} - \sqrt{\gamma_2})^2 \quad (4.7)$$

It is easy to verify the inequality

$$\sqrt{(1 - h_1\mu)(1 - h_2\mu)} \ll (1 - h_1\mu)^{-1}(1 - h_2\mu)^{-1} =$$
$$= \frac{1}{h_1 - h_2}\left(\frac{h_1}{1 - \mu h_2} - \frac{h_2}{1 - \mu h_1}\right) = \frac{1}{\sqrt{\gamma_1\gamma_2}}\sum_{k=0}^{\infty}\mu^k(h_1^{k+1} - h_2^{k+1}) \quad (4.8)$$

Computing directly in (4.6) the coefficients of the powers μ^0, μ^1 and dropping the terms with h_2 in (4.8), we obtain a series which dominates $\lambda_0(\mu)$, the solution of the equation (3.4)

$$\lambda_0(\mu) \ll \mu\gamma_2 + 2h_1(\gamma_1\gamma_2)^{-\frac{1}{2}}\sum_{k=2}^{\infty}(\mu h_1)^k, \qquad h_1 = 0.25(\sqrt{\gamma_1} + \sqrt{\gamma_2})^2 \quad (4.9)$$

We have thus obtained the following theorem.

Theorem 4.1. The expansion of the function $\lambda_0(\mu)$ in powers of μ, which determines the boundary of the zero region of instability ($\lambda_0(\mu) \to 0$ as $\mu \to 0$) of the solutions of the equation (1.1), is dominated by the series (4.9) which converges when

$$|\mu| \leqslant \mu_1 \equiv h_1^{-1} \equiv 4(\sqrt{\gamma_1} + \sqrt{\gamma_2})^{-2} \quad (4.10)$$

Note 4.1. Since $\gamma_1 > \gamma_2$ (4.2), it follows that $h^{-1} > \gamma_1^{-1}$. Therefore the series which determines $\lambda_0(\mu)$ converges when

$$|\mu| \leqslant \mu_2 \equiv \gamma_1^{-1} = \Big(\sum_{s=-\infty,\ s\neq 0}^{\infty} |a_s|\Big)^{-1} \quad (4.11)$$

Note 4.2. Let us introduce into (3.4) a new parameter $\lambda' = \lambda - \mu a_0$. Then one can construct a dominating function for $\Psi(\mu, \lambda)$ (3.4) which does not contain terms $O(\mu)$. This yields a new condition for convergence of the series that defines $\lambda_0(\mu)$

$$|\mu| < \mu_3 \equiv 4\gamma_1^{-1/2}(\sqrt{\gamma_1} + 2\sqrt{\gamma_2})^{-1}, \qquad \mu_3 > \mu_1 > \mu_2 \tag{4.12}$$

Note 4.3. From the convergence of the expansions of $\lambda_{n1}(\mu)$, $\lambda_{n2}(\mu)$ (1.3) follows the convergence of the power series in μ of the periodic solutions of the equation (1.1).

Example 4.1. Let us find the condition for convergence of the series defining $\lambda_0(\mu)$, the boundary of the zero region of instability of the equation of Mathieu ([6], pp. 18,25)

$$d^2y / dt^2 + (\lambda - 2\mu \cos 2t)\, y = 0 \tag{4.13}$$

From (1.2) we find $a_1 = a_{-1} = 1$. From (4.2) and from (4.10) to (4.12) we obtain

$$\gamma_1 = 2, \qquad \gamma_2 = 1, \qquad \mu_1 \approx 0.69, \qquad \mu_2 = 0.5, \qquad \mu_3 \approx 0.83 \tag{4.14}$$

The series for $\lambda_0(\mu)$ ([6], p.) is

$$\lambda_0(\mu) = -\tfrac{1}{2}\mu^2 + \tfrac{7}{128}\mu^4 - \tfrac{29}{2304}\mu^6 + \tfrac{68687}{18874368}\mu^8 + \ldots \tag{4.15}$$

which is known to converge when $|\mu| < 0.83$.

Example 4.2. For the differential equation

$$d^2y / dt^2 + [\lambda + \mu\,(1 + 2\cos 2t + 4\cos 4t)]\, y = 0 \tag{4.16}$$

we obtain by the method of a small parameter ([1], p.321) the equation

$$\lambda_0(\mu) = -\mu - \mu^2 + \varepsilon(\mu), \qquad \varepsilon(\mu) = O(\mu^3) \tag{4.17}$$

From (4.2), (4.7) and (1.2) we find

$$a_0 = -1, \quad a_1 = a_{-1} = -1, \quad a_2 = a_{-2} = -2, \quad \gamma_1 = 7, \quad \gamma_2 = 2, \quad h_1 \approx 4.07 \tag{4.18}$$

From (4.9) we obtain an estimate for $\varepsilon(\mu)$ when $|\mu| < 0.245$

$$|\varepsilon(\mu)| < \frac{2h_1}{\sqrt{\gamma_1\gamma_2}} \sum_{k=3}^{\infty} (\mu h_1)^k \approx \frac{159\,\mu^3}{1 - 4.07\mu} \tag{4.19}$$

5. Let us consider the case $n \neq 0$; $n = 1, 2, \ldots$. We will take the equation (2.2) in the form

$$y_k = \mu \sum_{s=-\infty}^{\infty}{}' d_n(k)\, a_{k-s}\, y_s + \mu d_n(k)\,(a_k y_0 + a_{k+n} y_{-n}) \tag{5.1}$$

The method of successive approximations [2] yields, when $k \neq 0, -n$

$$y_k = \mu\left[d_n(k)\, a_k + \mu \sum_{\alpha=-\infty}^{\infty}{}' d_n(k)\, a_{k-\alpha}\, d_n(\alpha)\, a_\alpha + \ldots\right] y_0 +$$
$$+ \mu\left[d_n(k)\, a_{k+n} + \mu \sum_{\alpha=-\infty}^{\infty}{}' d_n(k)\, a_{k-\alpha}\, d_n(\alpha)\, a_{\alpha+n} + \ldots\right] y_{-n} \tag{5.2}$$

Substituting $y_k (k \neq 0, -n)$ from (5.2) into the remaining two equations (2.2) when $k = 0, -n$, we obtain a system of two equations in two unknowns y_0, y_{-n},

$$f_1(\mu, \lambda)\, y_0 + f_2(\mu, \lambda)\, y_{-n} = 0, \qquad \bar f_2(\mu, \lambda)\, y_0 + \bar f_1(\mu, \lambda)\, y_{-n} = 0 \tag{5.3}$$

The bar above the letters indicates the complex conjugate. The functions f_1 and f_2 have the form

$$f_1(\mu, \lambda) = \lambda - n^2 + \mu a_0 + \mu^2 \sum_{\alpha=-\infty}^{\infty}{}' a_{-\alpha}\, d_n(\alpha)\, a_\alpha + \ldots$$
$$f_2(\mu, \lambda) = \mu a_n + \mu^2 \sum_{\alpha=-\infty}^{\infty}{}' a_{-\alpha}\, d_n(\alpha)\, a_{\alpha+n} + \ldots \tag{5.4}$$

The condition for the existence of a non-zero solution of the system (5.3) yields the equation of the boundary

$$|f_1(\mu, \lambda)|^2 - |f_2(\mu, \lambda)|^2 = 0 \tag{5.5}$$

Assuming that λ and μ are real, we introduce the notation, when $\lambda = z + n^2$

$$\begin{aligned} &\operatorname{Re} f_1(\mu, \lambda) = \lambda - n^2 + \mu a_0 + \mu^2 R_1(\mu, z), && \operatorname{Im} f_1(\mu, \lambda) = \mu^2 R_2(\mu, z) \\ &\operatorname{Re} f_2(\mu, \lambda) = \mu \operatorname{Re} a_n + \mu^2 R_3(\mu, z), && \operatorname{Im} f_2(\mu, \lambda) = \mu \operatorname{Im} a_n + \mu^2 R_4(\mu, z) \end{aligned} \tag{5.6}$$

The equation (5.5) solved for $z = \lambda - n^2$, yields

$$z = -\mu a_0 + \mu^2 R_1 \pm \sqrt{\mu^2 |a_n|^2 + 2\mu^3 (\operatorname{Re} a_n R_3 + \operatorname{Im} a_n R_4) + \mu^4 (R_3^2 + R_4^2 - R_2^2)} \tag{5.7}$$

It is necessary to determine the region of convergence of the series that determines the solution $\lambda_n(\mu)$ of the equation (5.7). We introduce an auxiliary lemma ([5], p.52).

Lemma 5.1. Suppose that $z(\mu)$ is an implicit function of μ defined by the equation

$$z = g_{10}\mu + g_{20}\mu^2 + g_{11}\mu z + g_{02} z^2 + \ldots \equiv \Psi(\mu, z) \tag{5.8}$$

where the right-hand side $\Psi(\mu, z)$ is a holomorphic and bounded function when

$$|\mu| < r, \qquad |z| < \rho, \qquad |\Psi(\mu, z)| < M \tag{5.9}$$

In this case $z(\mu)$ is given by a series which converges when

$$|\mu| < r^* = r\rho^2(\rho + 2M)^{-2} \tag{5.10}$$

and the series for z is dominated under the condition (5.10) by the series

$$\frac{\rho^2(\rho + 2M)^2}{8M(\rho + M)^2} \sum_{k=1}^{\infty} \left(\frac{\mu}{r^*}\right)^k \gg z(\mu) \tag{5.11}$$

Proof. Let us solve the auxiliary equation whose right-hand side dominates the function $\Psi(\mu, z)$ ([5], p. 52)

$$z = M(1 - \mu r^{-1})^{-1}(1 - z\rho^{-1})^{-1} - M - Mz\rho^{-1} \tag{5.12}$$

From (5.12) we have

$$\begin{aligned} z &= \frac{\rho^2}{2(\rho + M)}\left\{1 - \left[1 - \frac{\mu}{r}\left(\frac{\rho + 2M}{\rho}\right)^2\right]^{1/2}\left[1 - \frac{\mu}{r}\right]^{-1/2}\right\} \ll \\ &\ll \frac{\rho^2}{2(\rho + M)}\left\{\left[1 - \frac{\mu}{r}\left(\frac{\rho + 2M}{\rho}\right)^2\right]^{-1}\left[1 - \frac{\mu}{r}\right]^{-1} - 1\right\} \ll \\ &\ll \frac{\rho^2(\rho + 2M)^2}{8M(\rho + M)^2} \sum_{k=1}^{\infty}\left(\frac{\mu}{r}\right)^k\left(\frac{\rho + 2M}{\rho}\right)^{2k} \end{aligned} \tag{5.13}$$

The stated lemma follows from (5.13).

6. Let us consider the case $a_n \neq 0$, that is the case for which the width of the region of instability is $O(\mu)$. From (5.7) we obtain as a first approximation

$$\lambda_n = n^2 - \mu a_0 \pm \mu |a_n| + O(\mu^2), \qquad 0{,}5\,(b_1 - c_1) = |a| \neq 0 \tag{6.1}$$

Let us evaluate the functions $R_1, \ldots, R_4$ in (5.6). From (3.1) and (4.2) we have the inequalities for the powers of $z = \lambda - n^2$

$$d_n(k) = [\lambda - (n + 2k)^2]^{-1} = [\lambda - n^2 - 4n(n + k)]^{-1} \ll q(z) \tag{6.2}$$

where

$$z = \lambda - n^2, \qquad q(z) = (4 - z)^{-1}, \qquad |z| < 4 \tag{6.3}$$

Analogously to (4.4) we obtain from (5.4) and (5.6) the estimates

$$R_1 + iR_2 \ll \gamma_1\gamma_2 q(z)(1 - \mu\gamma_1 q(z))^{-1} \equiv R(\mu, z)$$
$$R_3 + iR_4 \ll \gamma_1\gamma_2 q(z)(1 - \mu\gamma_1 q(z))^{-1} \equiv R(\mu, z) \quad (6.4)$$

where, similarly to (4.2), we use the notation

$$\gamma_1 = \sum_{\substack{r=-\infty \\ s\neq 0,\, -n}}^{\infty} |a_s|, \qquad \gamma_2 = \max_s |a_s| \qquad (s = 0, \pm 1, \pm 2, \ldots) \quad (6.5)$$

The radical in (5.7) will be a holomorphic function of μ and z (6.3) in the region

$$|\mu| < r, \qquad |z| < \rho, \qquad 0 < \rho < 4 \quad (6.6)$$

if the following inequality is satisfied in the region (6.6):

$$|2\mu|a_n|^{-2}(\operatorname{Re} a_n R_3 + \operatorname{Im} a_n R_4) + \mu^2 |a_n|^{-2}(R_3^2 + R_4^2 - R_3^2)| < 1 \quad (6.7)$$

By means of the obvious inequality

$$a + bi|\cdot|c + di| = |(a + bi)(c - di)| = |ac + db + i(bc - ad)| \geqslant |ac + db| \quad (6.8)$$

and the notation $R(\mu, z)$ of (6.4), the inequality (6.7) can be transformed into the form

$$2r|a_n|^{-1}R(r, \rho) + r^2|a_n|^{-2}R^2(r, \rho) < 1 \quad (6.9)$$

or

$$r|a_n|^{-1}R(r, \rho) < \sqrt{2} - 1, \qquad 0 \leqslant \rho < 4 - r\gamma_1 [(\sqrt{2} + 1)\gamma_2 |a_n|^{-1} + 1] \quad (6.10)$$

When the inequality (6.10) is satisfied, the function represented by the root in (5.7) will be holomorphic in the region (6.6). Let us estimate the right-hand side of equation (5.7) in the regions (6.6) and (6.10). From (6.7) and (6.10) it follows that the next inequality is fulfilled in the region (6.6)

$$-\mu a_0 + \mu^2 R_1 \pm \{\mu^2|a_n|^2 + 2\mu^3(\operatorname{Re} a_n R_3 + \operatorname{Im} a_n R_4) + \mu^4(R_3^2 + R_4^2 - R_2^2)\}^{1/2}| \leqslant$$
$$\leqslant r|a_0| + r^2 R(r, \rho) + r|a_n|\sqrt{2} \leqslant r[|a_0| + (2\sqrt{2} - 1)|a_n|] \equiv M \quad (6.11)$$

We introduce the notation

$$\chi_1 = 0.25\gamma_1 [(\sqrt{2} + 1)\gamma_2 |a_n|^{-1} + 1], \qquad \chi_2 = 0.5[|a_0| + (2\sqrt{2} - 1)|a_n|] \quad (6.12)$$

In order to obtain, with the aid of Lemma 5.1, the largest value r^*

of the radius of convergence of the expansion $\lambda_n(\mu)$ (of the solution of (5.7)) in powers of μ, one must find

$$r^* = \max r\rho^2(\rho + 4\chi_2 r)^{-2}, \qquad 0 \leqslant \rho \leqslant 4(1 - r\chi_1), \qquad 0 \leqslant r \tag{6.13}$$

By the usual method we find that the maximum is attained when

$$r_0 = 2(2\chi_1 + \chi_2 + \sqrt{\chi_2^2 + 8\chi_1\chi_2})^{-1}, \qquad \rho_0 = 4(1 - \chi_1 r_0) \tag{6.14}$$

From Lemma 5.1, and from what has been said above, we deduce the next theorem.

Theorem 6.1. The expansions of $\lambda_{n1}(\mu)$ and $\lambda_{n2}(\mu)$, with $a_n \neq 0$ in (1.2), where a_n can be determined by the relation

$$a_n = 0.5 \lim_{\mu \to 0} \mu^{-1}(\lambda_{n1}(\mu) - \lambda_{n2}(\mu)) \neq 0 \tag{6.15}$$

converge when

$$|\mu| < r^* = r_0\rho_0^2(\rho_0 + 4\chi_2 r_0)^{-2} \tag{6.16}$$

where r_0, ρ_0 are determined in (6.14), (6.12) and (6.5). These expansions are dominated by the series

$$n^2 + \frac{\rho_0^2(\rho_0 + 4\chi_2 r_0)^2}{16\chi_2 r_0(\rho_0 + 2\chi_2 r_0)^2} \sum_{k=1}^{\infty} \left(\frac{\mu}{r^*}\right)^k \gg \lambda_{n1}(\mu), \lambda_{n2}(\mu) \tag{6.17}$$

Note 6.1. The radius of convergence r^* (6.16) may turn out to be considerably less than the actual radius of convergence.

Example 6.1. For Mathieu's equation (4.13), with $n = 1$, we have $|a_n| = 1 \neq 0$. From (6.5) we obtain $\gamma_1 = 1$, $\gamma_2 = 1$. From (6.12) we have $\chi_1 = 0.853$, $\chi_2 = 0.915$. From the formula (6.14) we find that $r_0 = 0.38$, $\rho_0 = 2.7$. Finally, from (6.16), the radius of convergence $r^* = 0.156$, while the dominating series (6.7) has the form

$$1 + 2\sum_{k=1}^{\infty}(6.4\mu)^k \gg \lambda_{1,1}(\mu), \lambda_{1,2}(\mu) \tag{6.18}$$

The actual expansion $\lambda_{1,1}(\mu)$, $\lambda_{1,2}(\mu)$ has the form ([6], p.25)

$$\lambda = 1 \pm \mu - \tfrac{1}{8}\mu^2 \pm \tfrac{1}{64}\mu^3 - \tfrac{1}{1536}\mu^4 \pm \tfrac{11}{36864}\mu^5 + O(\mu^6) \tag{6.19}$$

7. We shall now consider the last case for (1.1) when $a_n = 0$. From (6.1) it follows that

$$\lambda_{n1}(\mu) - \lambda_{n2}(\mu) = O(\mu^2) \qquad (l = 2, 3, \ldots) \tag{7.1}$$

Suppose that we know the first coefficients b_s, c_s $(s = 1, 2, \ldots, r)$ of the expansion (1.3), and that we have found that

$$b_1 - c_1 = 0, \ldots, b_{l-1} - c_{l-1} = 0, \qquad b_l - c_l = 2m > 0 \tag{7.2}$$

From the equations (5.7) and (7.2) it follows that

$$\sqrt{\mu^4 (R_3^2 + R_4^2 - R_2^2)} = m\mu^l + O(\mu^{l+1}) \tag{7.3}$$

We now introduce notations which make use of (5.6), $z = \lambda - n^2$

$$\Theta(\mu, z) = \mu^4 (R_3^2 + R_4^2 - R_2^2), \qquad \Pi(\mu, z) = \Theta(\mu, z) - \mu^{2l} m^2 \tag{7.4}$$

From (6.4) it follows that

$$\Theta(\mu, z) = \mu^4 [(R_3 + iR_4)(R_3 - iR_4) + R_2 R_2] \leqslant 2\mu^4 R^2(\mu, z) = \\ = 2\mu^4 \gamma_1^2 \gamma_2^2 q^2(z) \sum_{k=0}^{\infty} [\mu\gamma_1 q(z)]^k (k+1) \tag{7.5}$$

Since $\Pi(\mu, z) = O(\mu^{2l+1})$ when $\mu \to 0$, we see that

$$\Pi(\mu, z) \ll 2\mu^4 \gamma_1^2 \gamma_2^2 q^2(z) \sum_{k=2l-3}^{\infty} [\mu\gamma_1 q(z)]^k (k+1) = \\ = 2\mu^2 \gamma_2^2 [(2l-2)(\mu\gamma_1 q(z))^{2l-1} - (2l-3)(\mu\gamma_1 q(z))^{2l}] (1 - \mu\gamma_1 q(z))^{-2} \tag{7.6}$$

The expression $\sqrt{[\Theta(\mu, z)]}$ will be holomorphic in the region (6.6) if

$$|\Pi(r, \rho)| \leqslant \mu^{2l} m^2 \tag{7.7}$$

or

$$2\gamma_2^2 [(2l-2)(\gamma_1 q(\rho))^{2l-1} r - (2l-3)(\gamma_1 q(\rho))^{2l} r^2] \leqslant (1 - r\gamma_1 q(\rho))^2 m^2 \tag{7.8}$$

If we introduce the notation

$$\delta_1 = \gamma_1 q(\rho) + \gamma_2^2 (2l-2) [\gamma_1 q(\rho)]^{2l-1} m^{-2} \\ \delta_2 = \gamma_1^2 q(\rho) + 2\gamma_2^2 (2l-3) [\gamma_1 q(\rho)]^{2l} m^{-2} \tag{7.9}$$

then the inequality (7.8) will be satisfied if

$$r = (\delta_1 + \sqrt{\delta_1^2 - \delta_2})^{-1}, \qquad \delta_1^2 > \delta_2 \tag{7.10}$$

Let us evaluate the right-hand side of equation (5.7) in the region (6.6) taking into account (7.10) and the condition $a_n = 0$. We have

$$|-\mu a_0+\mu^2 R_1 \pm \sqrt{\Theta(\mu, z)}| \leqslant r|a_0|+r^2\gamma_1\gamma_2(4-\rho-r\gamma_1)^{-1}+mr^l\sqrt{2} \quad (7.11)$$

Introducing the quantity

$$M=r|a_0|+r^2\gamma_1\gamma_2(4-\rho-r\gamma_1)^{-1}+\sqrt{2}m^l \quad (7.12)$$

and applying Lemma 5.1, we obtain the next theorem.

Theorem 7.1. The expansions of $\lambda_{n1}(\mu)$ and $\lambda_{n2}(\mu)$ (1.3), with $a_n = 0$ and with the fulfillment of (7.2), converge if

$$|\mu|<r^*=r\rho^2(\rho+2M)^{-2} \quad (7.13)$$

and are dominated by the series

$$n^2+\frac{\rho^2(\rho+2M)^2}{8M(\rho+M)^2}\sum_{k=1}^{\infty}\left(\frac{\mu}{r^*}\right)^k \gg \lambda_{n1}(\mu), \qquad \lambda_{n2}(\mu) \quad (7.14)$$

For the computation of the quantities r, ρ and M we find the first non-zero coefficient $b_l - c_l$ in the expansion

$$\lambda_{n1}(\mu)-\lambda_{n2}(\mu)=(b_1-c_1)\mu+(b_2-c_2)\mu^2+\ldots+(b_n-c_n)\mu^n+\ldots \quad (7.15)$$

and set

$$m=0.5\,(b_l-c_l)\neq 0 \quad (7.16)$$

Let us take an arbitrary ρ, $0 < \rho < 4$. From (6.5) and (7.16) we compute the numbers δ_1 and δ_2 (7.9), and after that the quantities r (7.10) and M (7.12).

Example 7.1. For the equation of Mathieu (4.13) it has been found that $\lambda \approx 4$ ([6], p. 25)

$$\lambda_{2,1}(\mu)=4+\tfrac{5}{12}\mu^2-\ldots, \qquad \lambda_{2,2}(\mu)=4-\tfrac{1}{12}\mu^2+\ldots \quad (7.17)$$

From (7.15) and (7.16) we find $l = 2$, $m = 0.25$.

Let $\rho = 0.25$, $q(\rho) = 0.267$ (4.2). From (7.9) we have $\delta_1 \approx 6$, $\delta_2 \approx 3.3$. From the formula (7.10) we obtain $r = 0.0854$. The equation (7.12) implies that $M = 0.0066$. Finally, the radius of convergence (7.13) $r^* = 0.08$. The expansion (7.17) is known to be convergent when $|\mu| < 0.08$.

8. The above derived results are valid only in case the series that defines γ_1 (4.2), (6.5) converges. For a discontinuous function $a(t)$ in (1.1) this series will always diverge. One can extend the above obtained

theorems to the case when the function $a(t)$ (1.2) and its square are Lebesgue integrable on $[0, \pi]$ provided one considers the convergence of the series (3.3), (3.4), (5.2) and (5.4) as convergence in a Hilbert space l^2 ([3], p.92). Hence, we shall restrict ourselves to the extension of the Theorems 4.1, 6.1 and 7.1 to the case of a bounded function $a(t)$

$$|a(t)| \leqslant p, \qquad -\infty < t < +\infty, \qquad p = \text{const} \tag{8.1}$$

From (8.1) and from Bessel's inequality ([3], p.92), it follows that

$$|a_n| = \left|\frac{1}{\pi}\int_0^\pi a(t)\, e^{-2nit}\, dt\right| \leqslant p, \qquad \sum_{s=-\infty}^{\infty} |a_s|^2 \leqslant \frac{1}{\pi}\int_0^\pi |a(t)|^2\, dt \leqslant p^2 \tag{8.2}$$

Let us construct a series, which dominates (3.4) and (5.4), by making use of the inequality (8.2) alone. We shall have, for example, for (5.4)

$$\left|\sum_{\alpha=-\infty}^{\infty}{}' a_{-\alpha} d_n(\alpha)\, a_\alpha\right| \leqslant \left(\sum_{\alpha=-\infty}^{\infty}{}' |a_{-\alpha}|^2 d_n^2(\alpha)\right)^{1/2} \left(\sum_{\alpha=-\infty}^{\infty}{}' |a_\alpha|^2\right)^{1/2} \leqslant p^2 \varepsilon_n(z)$$

$$\left|\sum_{\alpha,\beta=-\infty}^{\infty}{}' a_{-\alpha} d_n(\alpha)\, a_{\alpha-\beta} d_n(\beta)\, a_\beta\right| \leqslant \left(\sum_{\alpha=-\infty}^{\infty}{}' |a_{-\alpha}|^2 d_n^2(\alpha)\right)^{1/2} \tag{8.3}$$

$$\left(\sum_{\beta=-\infty}^{\infty}{}' |a_{\alpha-\beta}|^2 d_n^2(\beta)\right)^{1/2} \left(\sum_{\beta=-\infty}^{\infty}{}' |a_\beta|^2\right)^{1/2} \leqslant p^3 \varepsilon_n^2(z) \text{ etc.}$$

Here $z = \lambda - n^2$

$$\varepsilon_n(z) = \left(\sum_{k=-\infty}^{\infty}{}' d_n^2(k)\right)^{1/2} = \left(\sum_{k=-\infty}^{\infty}{}' \frac{1}{[(n+2k)^2 - \lambda]^2}\right)^{1/2} =$$

$$= \left(\sum_{k=-\infty}^{\infty}{}' \frac{1}{[4k(n+k) - z]^2}\right)^{1/2} \leqslant \left(\sum_{k=-\infty}^{\infty}{}' \frac{(4-z)^{-2}}{k^2(n+k)^2}\right)^{1/2} = \eta_n\, q(z) \tag{8.4}$$

$$\eta_n = \left(\sum_{k=-\infty}^{\infty}{}' \frac{1}{k^2(n+k)^2}\right)^{1/2} \tag{8.5}$$

In particular, we obtain for η_n ($n = 0, 1, 2, \ldots$) the values (8.6)

$$\eta_0 = \left(2\sum_{k=1}^{\infty}\frac{1}{k^4}\right)^{1/2} \approx 1.47, \quad \eta_1 = \left(2\sum_{k=1}^{\infty}\frac{1}{k^2(k+1)^2}\right)^{1/2} < \left(0.5\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{1/2} \approx 1.28$$

For $n = 2, 3, 4, \ldots$

$$\eta_n = \left(2\sum_{k=1}^{\infty}\frac{1}{k^2(n+k)^2} + \sum_{k=-n+1}^{-1} k^2(k+n)^2\right)^{1/2} < \left(\frac{\pi^2}{3(n+1)^2} + \frac{1}{n-1}\right)^{1/2}$$

So that

$$\eta_2 < 1.17, \quad \eta_3 < 0.84, \quad \eta_4 < 0.68, \quad \eta_5 < 0.59 \quad \text{etc.}$$

From the evaluations (8.3) we find the dominating functions for R_1, ..., R_4 (5.6). Namely,

$$R_1 + iR_2 \ll p^2\varepsilon_n(1 - \mu p\varepsilon_n)^{-1}, \qquad R_3 + iR_4 \ll p^2\varepsilon_n(1 - \mu p\varepsilon_n)^{-1} \tag{8.7}$$

Comparing (8.7) and (6.4), we find that the estimates coincide if in (6.4) we set

$$\gamma_1 = p\eta_n, \qquad \gamma_2 = p \tag{8.8}$$

Since the remaining derivations coincide, we obtain a theorem which is a consequence of the Theorems 4.1, 6.1 and 7.1.

Theorem 8.1. Suppose that the function $a(t)$ of equation (1.1) satisfies the inequality (8.1), where $a(t)$ and its square are integrable on $[0, \pi]$.

1. The expansion, which determines the boundary of the zero region of instability of $\lambda_0(\mu)$, converges under the condition (4.10) and it is dominated by the series (4.9) where the quantities γ_1 and γ_2 are given by the formulas (8.8) and (8.6).

2. If $a_n \neq 0$ ($n = 1, 2, \ldots$), that is, condition (6.15) is fulfilled, then the expansions which determine the nth region of instability of $\lambda_{n1}(\mu)$, $\lambda_{n2}(\mu)$ (1.3) are dominated by the series (6.17) which converges under the condition (6.16). The quantities r_0 and ρ_0 are evaluated by means of the formulas (6.14), (8.8), (8.6) and (6.12). The formulas (6.12) can be replaced by the following ones:

$$\chi_1 = 0.25\,(2 + \sqrt{2})\,p^2\eta_n\,|a_n|^{-1}, \qquad \chi_2 = \sqrt{2}\,p \tag{8.9}$$

3. If it is known that

$$\lambda_{n1}(\mu) - \lambda_{n2}(\mu) = 2m\mu^l + O(\mu^{l+1}) \qquad (l = 2, 3, \ldots) \tag{8.10}$$

then the expansions $\lambda_{n1}(\mu)$, $\lambda_{n2}(\mu)$ are dominated by the series (7.14), and they converge if the condition (7.13) is fulfilled. The quantities r, ρ and M are evaluated in the same way as in Theorem 7.1 with the condition that the numbers γ_1 and γ_2 have already been determined by means of the formulas (8.8) and (8.6).

Example 8.1. Let us consider the differential equation

$$d^2y / dt^2 + (\lambda - \mu a(t))\,y = 0 \tag{8.11}$$

where $a(t)$ is a saw-shaped periodic function

$$a(t+\pi) \equiv a(t), \quad |a(t)| \leqslant p = \pi$$
$$a(t) = -\sum_{n=1}^{\infty} 2n^{-1} \sin 2nt, \quad a(t) = -\pi + 2t, \quad t \in [0, \pi] \tag{8.12}$$

From (8.8) we have

$$\gamma_1 = p\eta_0 \approx 4.62, \qquad \gamma_2 = p \approx 3.14 \tag{8.13}$$

The expression for $\lambda_0(\mu)$ converges if

$$|\mu| < \mu_1 \equiv 4(\sqrt{\gamma_1} + \sqrt{\gamma_2})^{-2} = 0.26 \tag{8.14}$$

Let us evaluate the radius of convergence r^* of the expansion $\lambda_{1,1}(\mu)$ and $\lambda_{1,2}(\mu)$. From the equation (8.9) we have $|a_n| = n^{-1}$, $|a_2| = 1$. The quantities χ_1 and χ_2 we obtain from (8.9), the quantities r_0 and ρ_0 from (6.14)

$$\chi_1 = 10.8, \quad \chi_2 = 4.44, \quad r_0 = 0.0416, \quad \rho_0 = 2.2 \tag{8.15}$$

Finally, from (6.16) we obtain the condition for convergence of the expansions $\lambda_{1,1}(\mu)$ and $\lambda_{1,2}(\mu)$ which determine the first region of instability

$$|\mu| < r^* = r_0\rho_0^2 (\rho_0 + 4\chi_2 r_0)^{-2} = 0.023 \tag{8.16}$$

BIBLIOGRAPHY

1. Malkin, I.G., *Nekotorye zadachi teorii nelineinykh kolebanii (Some problems of the theory of nonlinear oscillations)*. GITTL, 1956.

2. Valeev, K.G., K metodu Khilla v teorii lineinykh differentsial'nykh uravnenii s periodicheskimi koeffitsientami. Opredelenie kharakpericheskikh pokazatelei (On Hill's method in the theory of linear differential equations with periodic coefficients. Determination of the characteristic indices). *PMM* Vol. 25, No. 2, 1961.

3. Kantorovich, A.V. and Akilov, G.P., *Funktsional'nyi analiz v normirovannykh prostranstvakh (Functional analysis in normed spaces)*. GIFML, 14, 1959.

4. Filiminov, G.F., O vychislenii i isledovanii chastnykh reshenii obobshchenogo uravneniia Khilla (On the evaluation and investigation of particular solutions of the generalized equation of Hill). *PMM* Vol. 26, No. 3, 1962.

5. Erugin, N.P., *Neiavnye funktsii (Implicit functions)*. Izd-vo Leningr. un-ta, 1958.

6. Mak-Lakhlan, N.V., (McLachlan, N.W.), *Teoriia i prilozheniia funktsii Mat'e (Theory and applications of Mathieu functions). (Russian translation of English book)*. IL, 1953. Originally published by Clarendon Press, Oxford, 1947.

Translated by H.P.T.